

## Year 11 Maths · Unit 11.1.3 Quadratic Functions

**The big idea.** A quadratic function draws a parabola. Its equation can be written in three forms, and each form shows something different: the y-intercept, the x-intercepts or the vertex. Use the axis of symmetry, intercepts, domain, range and your GDC to describe the graph.

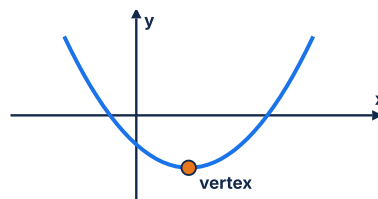
### Key vocabulary

| Word                      | Meaning                                                                   |
|---------------------------|---------------------------------------------------------------------------|
| <b>Quadratic function</b> | a function of the form $f(x) = ax^2 + bx + c$ with $a \neq 0$             |
| <b>Parabola</b>           | the curved graph of a quadratic function                                  |
| <b>Vertex</b>             | the turning point of a parabola: a minimum or a maximum                   |
| <b>Axis of symmetry</b>   | the vertical line through the vertex; the graph is the same on both sides |
| <b>y-intercept</b>        | the point where the graph crosses the y-axis ( $x = 0$ )                  |
| <b>x-intercept (zero)</b> | a point where the graph crosses the x-axis ( $y = 0$ )                    |
| <b>Standard form</b>      | $y = ax^2 + bx + c$                                                       |
| <b>Factorised form</b>    | $y = a(x - p)(x - q)$                                                     |
| <b>Vertex form</b>        | $y = a(x - h)^2 + k$                                                      |
| <b>Domain and range</b>   | the allowed inputs ( $x$ ) and the outputs ( $y$ ) of the function        |

### Quadratic functions and parabolas

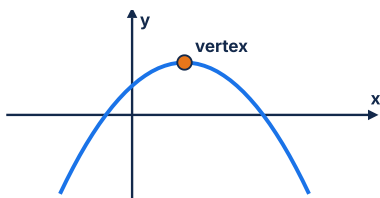
A **quadratic function** has the form  **$f(x)$**   
 **$= ax^2 + bx + c$** , where  $a \neq 0$ .

Its graph is a curve called a **parabola**.



If  **$a > 0$**  the parabola opens **upwards** (a U shape) and the vertex is a **minimum**.

**EXAMPLE**  $f(x) = x^2 - 4x + 5$



If  $a < 0$  the parabola opens **downwards** (a  $\cap$  shape) and the vertex is a **maximum**.

**EXAMPLE**  $f(x) = -x^2 + 2x + 3$

**CHECK 1** Does  $f(x) = 3 - 2x^2$  have a maximum or a minimum?

The size of  $a$  changes the width.

A bigger value of  $a$  (ignoring the sign) makes the parabola narrower.

**EXAMPLE**  $y = 3x^2$  is narrower than  $y = x^2$

**EXAMPLE**  $y = \frac{1}{2}x^2$  is wider than  $y = x^2$

## Intercepts and the axis of symmetry

The **y-intercept** is where  $x = 0$ .

In standard form it is the constant term  $c$ .

**EXAMPLE**  $f(x) = x^2 - 6x + 5$  has y-intercept  $(0, 5)$

The **x-intercepts** (zeros, roots) are where  $y = 0$ .

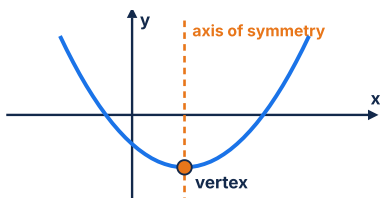
Solve the quadratic equation by factorising, using the formula or using your GDC.

**EXAMPLE**  $x^2 - 6x + 5 = 0$

**EXAMPLE**  $(x - 1)(x - 5) = 0$

**EXAMPLE**  $x = 1$  or  $x = 5$

**EXAMPLE** There can be 0, 1 or 2 x-intercepts



The **axis of symmetry** is a vertical line through the vertex.

Its equation is  $x = -\frac{b}{2a}$ .

It is halfway between the x-intercepts.

EXAMPLE  $f(x) = x^2 - 6x + 5$

EXAMPLE  $x = -\frac{-6}{2(1)} = 3$

EXAMPLE halfway between 1 and 5 is 3 ✓

The **vertex** lies on the axis of symmetry.

Substitute the axis value into  $f(x)$  to find the y-coordinate.

EXAMPLE  $f(3) = 9 - 18 + 5 = -4$

EXAMPLE vertex (3, -4)

**CHECK 2** Find the axis of symmetry of  $f(x) = x^2 + 8x + 3$

## The three forms of a quadratic

**Standard form:**  $y = ax^2 + bx + c$ .

It shows the **y-intercept**  $c$ .

EXAMPLE  $y = x^2 - 6x + 5$

**Factorised form:**  $y = a(x - p)(x - q)$ .

It shows the **x-intercepts**  $p$  and  $q$ .

EXAMPLE  $y = (x - 1)(x - 5)$

EXAMPLE x-intercepts at 1 and 5

EXAMPLE Beware the signs:  $(x + 2)$  gives an x-intercept of  $-2$

**Vertex form:**  $y = a(x - h)^2 + k$ .

It shows the **vertex**  $(h, k)$  and the axis of symmetry  $x = h$ .

EXAMPLE  $y = (x - 3)^2 - 4$

EXAMPLE vertex (3, -4)

EXAMPLE Beware the sign:  $(x + 2)^2$  gives  $h = -2$

All three forms describe the **same** graph.

The value of  $a$  is the same in each form.

EXAMPLE  $y = x^2 - 6x + 5 = (x - 1)(x - 5) = (x - 3)^2 - 4$

## Converting between the forms

**Factorised to standard:** expand the brackets.

| **EXAMPLE**  $y = (x - 1)(x - 5)$

| **EXAMPLE**  $= x^2 - 5x - x + 5$

| **EXAMPLE**  $= x^2 - 6x + 5$

**Standard to factorised:** solve  $ax^2 + bx + c = 0$  to find  $p$  and  $q$ , then write  $y = a(x - p)(x - q)$ .

| **EXAMPLE**  $y = x^2 - 6x + 5$

| **EXAMPLE** zeros 1 and 5

| **EXAMPLE**  $y = (x - 1)(x - 5)$

**Standard to vertex:** find  $h = -\frac{b}{2a}$ , then  $k = f(h)$ , then write  $y = a(x - h)^2 + k$ .

| **EXAMPLE**  $y = 2x^2 - 12x + 13$

| **EXAMPLE**  $h = -\frac{-12}{4} = 3$

| **EXAMPLE**  $k = 2(9) - 36 + 13 = -5$

| **EXAMPLE**  $y = 2(x - 3)^2 - 5$

**Vertex to standard:** expand the bracket and collect terms.

| **EXAMPLE**  $y = 2(x - 3)^2 - 5$

| **EXAMPLE**  $= 2(x^2 - 6x + 9) - 5$

| **EXAMPLE**  $= 2x^2 - 12x + 13$

**Factorised to vertex:** the axis is  $x = \frac{p + q}{2}$ , so  $h = \frac{p + q}{2}$ . Substitute  $h$  to find  $k$ .

| **EXAMPLE**  $y = (x - 1)(x - 5)$

| **EXAMPLE**  $h = 3$  and  $k = (2)(-2) = -4$

| **EXAMPLE**  $y = (x - 3)^2 - 4$

**CHECK 3** Write  $y = (x - 2)(x + 4)$  in standard form

## Which form to use

Use **standard form** to read the  $y$ -intercept and to type into a calculator.

| **EXAMPLE** the  $y$ -intercept of  $y = x^2 - 6x + 5$  is 5

Use **factorised form** to read the  $x$ -intercepts (zeros) straight away.

| **EXAMPLE**  $y = (x - 1)(x - 5)$  crosses the  $x$ -axis at 1 and 5

Use **vertex form** to read the maximum or minimum and where it happens.

**EXAMPLE**  $y = (x - 3)^2 - 4$  has a minimum value of  $-4$  at  $x = 3$

Vertex form also shows the **transformations** of  $y = x^2$ .

$y = a(x - h)^2 + k$  is a stretch of factor  $a$ , then a translation of  $h$  units right and  $k$  units up.

**EXAMPLE**  $y = (x - 3)^2 - 4$  is  $y = x^2$  moved 3 right and 4 down

## Domain and range

The **domain** of a quadratic function is all real numbers,  $x \in \mathbb{R}$ , unless the question restricts it.

The **range** depends on the vertex.

If  $a > 0$ :  $y \geq k$ .

If  $a < 0$ :  $y \leq k$ .

Here  $k$  is the  $y$ -coordinate of the vertex.

**EXAMPLE**  $y = (x - 3)^2 - 4$  has range  $\{y \in \mathbb{R} : y \geq -4\}$

**EXAMPLE**  $y = 5 - x^2$  has range  $\{y \in \mathbb{R} : y \leq 5\}$

With a **restricted domain**, check the vertex and both end points.

**EXAMPLE**  $f(x) = x^2 - 6x + 5$  for  $0 \leq x \leq 4$

**EXAMPLE**  $f(0) = 5$ ,  $f(4) = -3$ , vertex  $(3, -4)$

**EXAMPLE** range  $\{y \in \mathbb{R} : -4 \leq y \leq 5\}$

**CHECK 4** State the range of  $f(x) = x^2 + 2$

## Sketching a quadratic by hand

Step 1: decide the shape from the sign of  $a$ .

**EXAMPLE** Step 2: find the  $y$ -intercept ( $x = 0$ ).

**EXAMPLE** Step 3: find the  $x$ -intercepts ( $y = 0$ ), if there are any.

Step 4: find the axis of symmetry and the vertex.

Step 5: draw a smooth curve through the points, symmetrical about the axis.

Label the intercepts and the vertex with coordinates.

**EXAMPLE** The point  $(0, 5)$  on  $y = x^2 - 6x + 5$  has a mirror point  $(6, 5)$

## GDC skills: TI-Nspire

**Graph it:** open a **Graphs** page and type the function into  $f1(x)$ .

Press **enter**.

Adjust the window with **menu** → **Window / Zoom** until you can see the intercepts and the vertex.

**Features of the graph:** **menu** → **Analyze Graph**.

Choose **Zero** ( $x$ -intercepts), **Minimum** or **Maximum** (vertex).

Move the pointer to the left of the point, press **enter**, then to the right and press **enter**.

**Table of values:** **menu** → **Table** → **Split-screen Table** (or **ctrl + T**).

Use it to check values and to see the symmetry.

**Intersections:** type a second function into  $f2(x)$ , then **menu** → **Analyze Graph** → **Intersection**.

The  $x$ -values of the intersections solve  $f1(x) = f2(x)$ .

**EXAMPLE**  $x^2 = 2x + 3$  gives  $x = -1$  and  $x = 3$

**Domain and range on the GDC:** set the window to the domain you need, then read the range from the minimum, maximum or end points.

Give answers to 3 significant figures unless told otherwise, and write down what each point means.

## Using quadratic functions to model

A ball is thrown upwards. Its height  $h$  metres after  $t$  seconds is  $h(t) = -5t^2 + 20t$ .

**EXAMPLE** The vertex is at  $t = -$

$$\frac{20}{2(-5)} = 2$$

**EXAMPLE**  $h(2) = -20 + 40 = 20$  so the maximum height is 20 m

When does the ball land?

**EXAMPLE**  $h(t) = 0$

**EXAMPLE**  $-5t(t - 4) = 0$

**EXAMPLE**  $t = 0$  or  $t = 4$

**EXAMPLE** the ball lands after 4 seconds

**CHECK 5** What are the domain and range in this problem?

Interpret every answer in context.  
Reject answers that do not make sense, such as negative times.

## Common mistakes

×

“The vertex of  $y = (x - 3)^2 + 4$  is  $(-3, 4)$ .”

✓

The sign changes: the vertex is  $(3, 4)$ .

×

“The axis of symmetry is  $y = -$

$$\frac{b}{2a}$$

.”

✓

It is a vertical line:  $x = -$

$$\frac{b}{2a}$$

.

×

“The y-intercept is where  $y = 0$ .”

✓

The y-intercept is where  $x = 0$ . The x-intercepts are where  $y = 0$ .

×

“The range of  $f(x) = x^2 + 2$  is all real numbers.”

✓

The domain is all real numbers. The range is  $\{y \in \mathbb{R} : y \geq 2\}$ .

×

“If  $a > 0$  the vertex is a maximum.”

✓

If  $a > 0$  the parabola opens upwards, so the vertex is a minimum.

×

“The x-intercepts of  $y = (x - 2)(x + 5)$  are 2 and 5.”

✓

They are 2 and  $-5$ . Change the sign inside the bracket.

×

“With a restricted domain, the range is found from the vertex only.”

✓

Also check both end points of the domain.

## Answers to the checks

Cover this page until you have tried the check yourself.

| Check   | Answer                                                                                                                 |
|---------|------------------------------------------------------------------------------------------------------------------------|
| Check 1 | Does $f(x) = 3 - 2x^2$ have a maximum or a minimum?<br><b>Answer:</b> A maximum, because $a = -2$ is negative.         |
| Check 2 | Find the axis of symmetry of $f(x) = x^2 + 8x + 3$<br><b>Answer:</b> $x = -\frac{8}{2} = -4$                           |
| Check 3 | Write $y = (x - 2)(x + 4)$ in standard form<br><b>Answer:</b> $x^2 + 4x - 2x - 8 = x^2 + 2x - 8$                       |
| Check 4 | State the range of $f(x) = x^2 + 2$<br><b>Answer:</b> $\{y \in \mathbb{R} : y \geq 2\}$                                |
| Check 5 | What are the domain and range in this problem?<br><b>Answer:</b> Domain $0 \leq t \leq 4$ . Range $0 \leq h \leq 20$ . |