

Year 11 Maths · Unit 11.1.2 Solving Quadratic Equations

The big idea. Every quadratic equation can be solved, and the method depends on the equation: factorising, the quadratic formula, and, in the extended class, completing the square and the discriminant.

Key vocabulary

Word	Meaning
Quadratic equation	an equation that can be written $ax^2 + bx + c = 0$ with $a \neq 0$
Root (zero)	a solution of an equation; where a graph crosses the x-axis
Factorise	write an expression as a product of brackets
Zero product rule	if $p \times q = 0$ then $p = 0$ or $q = 0$
Difference of two squares	$a^2 - b^2 = (a + b)(a - b)$
Quadratic formula	the formula that solves any quadratic equation $ax^2 + bx + c = 0$
Completing the square	rewriting $x^2 + bx + c$ as $(x + \text{half of } b)^2$ plus a number
Discriminant	$\Delta = b^2 - 4ac$, which tells you how many real solutions there are
Real solution	a solution that is a real number (one on the number line)

Quadratic equations and the zero product rule

A **quadratic equation** can be written as **$ax^2 + bx + c = 0$** , where $a \neq 0$.
The solutions are called **roots** or **zeros**.
There are at most two real roots.

EXAMPLE $x^2 - 5x + 6 = 0$

Always rearrange so that one side is 0 before you factorise.

EXAMPLE $x^2 = 5x$ becomes $x^2 - 5x = 0$

CHECK 1 Rearrange $x^2 + 3x = 10$ so that one side is 0

Zero product rule: if $p \times q = 0$ then $p = 0$ or $q = 0$.

One of the factors must be zero.

EXAMPLE $(x - 2)(x + 5) = 0$

EXAMPLE $x - 2 = 0$ or $x + 5 = 0$

EXAMPLE $x = 2$ or $x = -5$

Never divide both sides by x .

You lose the solution $x = 0$.

EXAMPLE $x^2 = 5x$

EXAMPLE $x^2 - 5x = 0$

EXAMPLE $x(x - 5) = 0$

EXAMPLE $x = 0$ or $x = 5$

Solving by factorising: common factors

Take out the **highest common factor**, including x.

| **EXAMPLE** $x^2 + 6x = 0$

| **EXAMPLE** $x(x + 6) = 0$

| **EXAMPLE** $x = 0$ or $x = -6$

The common factor can include a number.

| **EXAMPLE** $3x^2 - 12x = 0$

| **EXAMPLE** $3x(x - 4) = 0$

| **EXAMPLE** $x = 0$ or $x = 4$

CHECK 2 Solve $x^2 - 7x = 0$

Check each answer by substituting.

| **EXAMPLE** $x = 4$: $3(16) - 12(4) = 48 - 48 = 0$ ✓

Solving by factorising: difference of two squares

$a^2 - b^2 = (a + b)(a - b)$.

Use it when there are two squared terms with a minus between them and no x term.

| **EXAMPLE** $x^2 - 49 = 0$

| **EXAMPLE** $(x + 7)(x - 7) = 0$

| **EXAMPLE** $x = 7$ or $x = -7$

The first term can have a coefficient.

| **EXAMPLE** $4x^2 - 25 = 0$

| **EXAMPLE** $(2x + 5)(2x - 5) = 0$

| **EXAMPLE** $x = -\frac{5}{2}$ or $x = \frac{5}{2}$

CHECK 3 Solve $x^2 - 81 = 0$

$x^2 + 49$ has **no** real factors, because a plus sign gives no real roots. $x^2 = 49$ has **two** solutions: $x = \pm 7$.

Solving by factorising: product and sum ($a = 1$)

For $x^2 + bx + c$, find two numbers with **product c** and **sum b** .

EXAMPLE $x^2 + 5x + 6 = 0$

EXAMPLE product 6 and sum 5: 2 and 3

EXAMPLE $(x + 2)(x + 3) = 0$

EXAMPLE $x = -2$ or $x = -3$

If c is **negative**, the two numbers have different signs.

If c is **positive**, they have the same sign as b .

EXAMPLE $x^2 - 2x - 15 = 0$

EXAMPLE product -15 and sum -2 : -5 and 3

EXAMPLE $(x - 5)(x + 3) = 0$

EXAMPLE $x = 5$ or $x = -3$

Check by expanding the brackets.

EXAMPLE $(x - 5)(x + 3) = x^2 + 3x - 5x - 15 = x^2 - 2x - 15 \checkmark$

CHECK 4 Solve $x^2 + 7x + 12 = 0$

Solving by factorising: $a \neq 1$

Multiply $a \times c$, then find two numbers with that product and sum b .

Split the middle term and factorise in pairs.

EXAMPLE $2x^2 + 7x + 3 = 0$

EXAMPLE $a \times c = 6$, and $1 + 6 = 7$

EXAMPLE $2x^2 + x + 6x + 3 = 0$

EXAMPLE $x(2x + 1) + 3(2x + 1) = 0$

EXAMPLE $(2x + 1)(x + 3) = 0$

Then use the zero product rule.

EXAMPLE $2x + 1 = 0$ or $x + 3 = 0$

EXAMPLE $x = -\frac{1}{2}$ or $x = -3$

Another example.

EXAMPLE $6x^2 - 5x - 6 = 0$

EXAMPLE $a \times c = -36$, and $-9 + 4 = -5$

EXAMPLE $6x^2 - 9x + 4x - 6 = 0$

EXAMPLE $3x(2x - 3) + 2(2x - 3) = 0$

EXAMPLE $(2x - 3)(3x + 2) = 0$

EXAMPLE $x = \frac{3}{2}$ or $x = -\frac{2}{3}$

CHECK 5 Solve $3x^2 + 10x + 8 = 0$

Take out a common factor first when you can.

EXAMPLE $4x^2 + 10x + 6 = 0$

EXAMPLE $2(2x^2 + 5x + 3) = 0$

EXAMPLE $2(2x + 3)(x + 1) = 0$

The quadratic formula

For $ax^2 + bx + c = 0$: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

It works for **every** quadratic, even when the equation will not factorise.

EXAMPLE $\pm$ gives two answers: one with + and one with -

Step 1: rearrange to $= 0$ and write down a , b and c , with their signs.

Step 2: substitute using brackets.

EXAMPLE $2x^2 + 3x - 4 = 0$

EXAMPLE $a = 2$, $b = 3$, $c = -4$

Substitute into the formula.

EXAMPLE $x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-4)}}{2(2)}$

EXAMPLE $x = \frac{-3 \pm \sqrt{41}}{4}$

Step 3: work out both answers, then round if asked.

Give exact answers unless the question asks for rounding.

EXAMPLE $x = \frac{-3 + \sqrt{41}}{4} = 0.851$ (3 s.f.)

EXAMPLE $x = \frac{-3 - \sqrt{41}}{4} = -2.35$ (3 s.f.)

A negative b needs brackets.

EXAMPLE $x^2 - 4x + 1 = 0$

EXAMPLE $a = 1, b = -4, c = 1$

EXAMPLE $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$

EXAMPLE $x = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$

Always start by naming a, b and c.

Keep the sign with each number.

CHECK 6 What are a, b and c in $3x^2 - 5x + 1 = 0$

Which method should I use?

Rearrange to $= 0$ first.

Then look for the pattern.

EXAMPLE common factor: no constant term ($x^2 + 6x$)

EXAMPLE difference of two squares: $x^2 - 49$

If there is a middle term and integers work, **factorise** (product and sum, or split the middle term).

Check by expanding.

If it does not factorise easily, use the **quadratic formula**.

The formula always works.

On a calculator paper, check with your GDC (graph, zeros).

A problem in context may give a root that makes no sense.

EXAMPLE A rectangle is x by $(x + 3)$ with area 40

EXAMPLE $x^2 + 3x - 40 = 0$

EXAMPLE $(x + 8)(x - 5) = 0$

EXAMPLE $x = 5$ (reject $x = -8$, a length cannot be negative)

Extended: solving by completing the square

Write $x^2 + bx$ as $(x + \frac{b}{2})^2 - (\frac{b}{2})^2$.

Add and subtract the square of half the coefficient of x.

EXAMPLE $x^2 + 6x + 2 = 0$

EXAMPLE $(x + 3)^2 - 9 + 2 = 0$

EXAMPLE $(x + 3)^2 = 7$

Take the square root of **both** sides, with $\pm$.

EXAMPLE $x + 3 = \pm\sqrt{7}$

EXAMPLE $x = -3 \pm \sqrt{7}$

If $a \neq 1$, take out a factor from the x terms first.

EXAMPLE $2x^2 + 8x + 1 = 0$

EXAMPLE $2[(x + 2)^2 - 4] + 1 = 0$

EXAMPLE $2(x + 2)^2 = 7$

Then divide and take the square root.

EXAMPLE $(x + 2)^2 = \frac{7}{2}$

EXAMPLE $x = -2 \pm \sqrt{\frac{7}{2}}$

If b is odd, the half is a fraction.

The answer matches the quadratic formula.

EXAMPLE $x^2 + 3x - 1 = 0$

EXAMPLE $(x + \frac{3}{2})^2 - \frac{9}{4} - 1 = 0$

EXAMPLE $(x + \frac{3}{2})^2 = \frac{13}{4}$

EXAMPLE $x = \frac{-3 \pm \sqrt{13}}{2}$

Use the same steps every time.

CHECK 7 Solve $x^2 + 4x - 3 = 0$ by completing the square

The form $(x - h)^2 + k$ also shows the **vertex** (h, k) of the graph.

This links to quadratic functions.

Extended: the discriminant

The **discriminant** is $\Delta = b^2 - 4ac$. It is the part under the square root in the formula.

It tells you how many real solutions there are, without solving.

EXAMPLE $x^2 + 2x - 3: \Delta = 4 + 12 = 16$

$\Delta > 0$: two different real solutions.

$\Delta = 0$: one repeated solution.

$\Delta < 0$: no real solutions.

EXAMPLE $x^2 + 4x + 4: \Delta = 16 - 16 = 0$, so $x = -2$ only

EXAMPLE $x^2 + 2x + 5: \Delta = 4 - 20 = -16 < 0$, so no real solutions

Link to the graph: $\Delta > 0$ crosses the x-axis twice.

$\Delta = 0$ touches it once at the vertex.

$\Delta < 0$ does not reach the x-axis.

CHECK 8 How many real solutions does $x^2 - 3x + 5 = 0$ have?

Find an unknown from the condition on Δ .

EXAMPLE $x^2 + 6x + k = 0$ has one solution

EXAMPLE $36 - 4k = 0$

EXAMPLE $k = 9$

EXAMPLE For two solutions: $36 - 4k > 0$, so $k < 9$

Common mistakes

✗

" $(x - 3)(x + 2) = 6$, so $x - 3 = 6$ or $x + 2 = 6$."

✓

The zero product rule needs 0 on one side. Expand: $x^2 - x - 12 = 0$, then $(x - 4)(x + 3) = 0$, so $x = 4$ or $x = -3$.

✗

" $x^2 = 9$, so $x = 3$."

✓

There are two solutions: $x = 3$ or $x = -3$.

✗

" $x^2 = 5x$, so $x = 5$."

✓

Dividing by x loses a solution. $x^2 - 5x = 0$ gives $x(x - 5) = 0$, so $x = 0$ or $x = 5$.

✗

"In the formula, $-b = -5$ when $b = -5$."

✓

$-b = -(-5) = 5$. Put negative numbers in brackets.

✗

"Only the square root is divided by $2a$."

✓

The whole numerator, $-b \pm \sqrt{b^2 - 4ac}$, is divided by $2a$.

✗

" $(x + 3)^2 = 7$, so $x + 3 = \sqrt{7}$."

✓

Use $\pm$: $x + 3 = \pm\sqrt{7}$, so $x = -3 \pm \sqrt{7}$.

✗

" $\Delta < 0$ means the equation has no solutions at all."

✓

It has no real solutions, and the graph does not cross the x-axis.

Answers to the checks

Cover this page until you have tried the check yourself.

Check	Answer
Check 1	Rearrange $x^2 + 3x = 10$ so that one side is 0 Answer: $x^2 + 3x - 10 = 0$
Check 2	Solve $x^2 - 7x = 0$ Answer: $x(x - 7) = 0$, so $x = 0$ or $x = 7$.
Check 3	Solve $x^2 - 81 = 0$ Answer: $(x + 9)(x - 9) = 0$, so $x = 9$ or $x = -9$.
Check 4	Solve $x^2 + 7x + 12 = 0$ Answer: 3 and 4: $(x + 3)(x + 4) = 0$, so $x = -3$ or $x = -4$.
Check 5	Solve $3x^2 + 10x + 8 = 0$ Answer: $a \times c = 24$, $4 + 6 = 10$: $(3x + 4)(x + 2) = 0$, so $x = -\frac{4}{3}$ or $x = -2$.
Check 6	What are a, b and c in $3x^2 - 5x + 1 = 0$ Answer: $a = 3$, $b = -5$, $c = 1$
Check 7	Solve $x^2 + 4x - 3 = 0$ by completing the square Answer: $(x + 2)^2 = 7$, so $x = -2 \pm \sqrt{7}$.
Check 8	How many real solutions does $x^2 - 3x + 5 = 0$ have? Answer: $\Delta = 9 - 20 = -11 < 0$, so none.