

Year 11 Maths · Unit 11.1.1 Functions

The big idea. A function gives exactly one output for every input. Learn the language (set notation, domain, range, function notation), then, if you are in the extended class, inverse and composite functions.

Key vocabulary

Word	Meaning
Set	a collection of items, called elements
Element	an item in a set; $\in$ means “is an element of”
Relation	any set of ordered pairs (input, output)
Function	a relation where every input has exactly one output
Domain	the set of all allowed inputs (x values)
Range	the set of all outputs (y values)
Function notation	$f(x)$ is read “f of x”: the output when the input is x
Vertical line test	a graph is a function if every vertical line crosses it at most once
Inverse function	f^{-1} undoes f: if $f(a) = b$ then $f^{-1}(b) = a$
Composite function	$(f \circ g)(x) = f(g(x))$: do g first, then f

Prerequisite: solving linear equations

Do the **same thing to both sides** to keep the equation balanced.

Undo addition and subtraction first, then undo multiplication and division.

EXAMPLE $3x + 5 = 20$

EXAMPLE $3x = 15$

EXAMPLE $x = 5$

CHECK 1 Solve $4x - 7 = 21$

Brackets: expand them, or divide both sides by the number outside.

EXAMPLE $2(x - 3) = 10$

EXAMPLE $x - 3 = 5$

EXAMPLE $x = 8$

CHECK 2 Solve $3(x + 2) = 21$

Unknowns on both sides: collect the x terms on one side.

EXAMPLE $5x - 4 = 2x + 11$

EXAMPLE $3x = 15$

EXAMPLE $x = 5$

CHECK 3 Solve $7x + 1 = 3x + 13$

Fractions: multiply both sides to clear the denominator.

EXAMPLE $\frac{x}{4} + 3 = 5$

EXAMPLE $x = 8$

CHECK 4 Solve $\frac{x}{3} - 2 = 4$

Set notation

A **set** is a collection of items. Each item is an **element**. The symbol $\in$ means “is an element of”.

EXAMPLE $3 \in \{1, 2, 3\}$

EXAMPLE $5 \notin \{1, 2, 3\}$

Number sets: $\mathbb{R}$ all real numbers. $\mathbb{Z}$ the integers ($\dots, -2, -1, 0, 1, 2, \dots$). $\mathbb{Q}$ the rational numbers. $\mathbb{Z}^+$ the positive integers ($1, 2, 3, \dots$).

$\sqrt{2} \in \mathbb{R}$ but $\sqrt{2} \notin \mathbb{Q}$

EXAMPLE $-4 \in \mathbb{Z}$

Set-builder notation: $\{x \in \mathbb{R} : x > 2\}$.

Read it as “the set of all real numbers x such that x is greater than 2”.

EXAMPLE 2 is not in this set (the inequality is strict)

EXAMPLE 2.1 is in this set

Inequality signs: $<$ and $>$ leave the end value out. $\leq$ and $\geq$ include it.

EXAMPLE $\{x \in \mathbb{R} : -1 \leq x < 4\}$ includes -1 but not 4

CHECK 5 Is 4 in $\{x \in \mathbb{R} : x \geq 4\}$?

Relations and functions

A **relation** is any set of ordered pairs (input, output).

EXAMPLE $\{(1, 2), (2, 4), (3, 6)\}$

A **function** is a relation where each input has **exactly one** output.

The same output can come from different inputs.

EXAMPLE $\{(1, 2), (2, 4), (3, 4)\}$ is a function

EXAMPLE $\{(1, 2), (1, 5)\}$ is not (input 1 gives two outputs)

Types of mapping: **one-to-one** and **many-to-one** are functions.

One-to-many is not a function.

| **EXAMPLE** $y = 2x$ is one-to-one

| **EXAMPLE** $y = x^2$ is many-to-one

CHECK 6 Is $\{(2, 3), (4, 3), (5, 1)\}$ a function?

A function is a rule: put x in, get exactly one y out.

Think of a machine.

The input goes in, the rule is applied, and one output comes out.

Domain and range

Domain is the set of all allowed **inputs** (x values).

Range is the set of all **outputs** (y values).

Domain comes before range, and x comes before y .

| **EXAMPLE** $\{(1, 3), (2, 5), (4, 5)\}$ has domain $\{1, 2, 4\}$ and range $\{3, 5\}$

A **restricted domain** limits the inputs.

| **EXAMPLE** $f(x) = 2x + 1$ with $x \in \{0, 1, 2, 3\}$

| **EXAMPLE** Range is $\{1, 3, 5, 7\}$

CHECK 7 Find the range of $f(x) = x + 4$ for $x \in \{1, 2, 3\}$

Some inputs are not allowed.

You cannot divide by zero and you cannot take the square root of a negative number.

| **EXAMPLE** $f(x) = \frac{1}{x}$ has domain $\{x \in \mathbb{R} : x \neq 0\}$

| **EXAMPLE** $f(x) = \sqrt{x}$ has domain $\{x \in \mathbb{R} : x \geq 0\}$

Unrestricted domain is all real numbers, written $x \in \mathbb{R}$.

The range depends on the shape of the graph.

| **EXAMPLE** $f(x) = x^2$ has domain $\mathbb{R}$

| **EXAMPLE** range $\{y \in \mathbb{R} : y \geq 0\}$

CHECK 8 State the domain of $f(x) = \frac{1}{x - 3}$

Graphs and the vertical line test

Vertical line test: a graph shows a function if every vertical line crosses it **at most once**.

If any vertical line crosses twice, one input has two outputs.

EXAMPLE $y = x^2$ passes

EXAMPLE a circle fails

EXAMPLE a sideways parabola fails

Reading the **domain** from a graph: look **left to right** along the x-axis.

Reading the **range**: look **bottom to top** along the y-axis.

Open circle ○: the end value is not included. **Closed circle** ●: the end value is included.

EXAMPLE a closed circle at $x = 2$ gives $x \leq 2$

EXAMPLE an open circle at $y = 5$ gives $y < 5$

Check the lowest and highest points, and check for gaps or breaks.

EXAMPLE $y = x^2 - 4$ has lowest point -4 , so range $\{y \in \mathbb{R} : y \geq -4\}$

Function notation

f(x) is read “f of x”.

It is the output when the input is x.

It does **not** mean f times x.

EXAMPLE $f(x) = 3x - 2$

EXAMPLE $f(4) = 3(4) - 2 = 10$

To **evaluate** a function, replace every x with the input in brackets.

EXAMPLE $f(x) = x^2 + 1$

EXAMPLE $f(-3) = (-3)^2 + 1 = 9 + 1 = 10$

CHECK 9 If $f(x) = 5 - 2x$, find $f(3)$

To find the input for a given output, **solve an equation**.

EXAMPLE $f(x) = 3x - 2$ and $f(x) = 13$

EXAMPLE $3x - 2 = 13$

EXAMPLE $x = 5$

CHECK 10 If $f(x) = 2x + 3$, solve $f(x) = 11$

Other letters are used too (g, h). $y = f(x)$ means the same as $f(x)$.

The point $(a, f(a))$ is on the graph.

EXAMPLE $f(2) = 7$ means the point $(2, 7)$ is on the graph

Extended: inverse functions

The **inverse function** f^{-1} undoes f .

If $f(a) = b$ then $f^{-1}(b) = a$.

$f^{-1}(x)$ does **not** mean $\frac{1}{f(x)}$.

EXAMPLE $f(x) = x + 3$ and $f^{-1}(x) = x - 3$

To find f^{-1} : write $y = f(x)$, **swap x and y** , then rearrange to make y the subject.

EXAMPLE $f(x) = 3x - 2$

EXAMPLE $x = 3y - 2$

EXAMPLE $3y = x + 2$

EXAMPLE $f^{-1}(x) = \frac{x + 2}{3}$

Check with a number or with composition.

EXAMPLE $f(4) = 10$ and $f^{-1}(10) = \frac{10 + 2}{3} = 4$

EXAMPLE $f(f^{-1}(x)) = x$

CHECK 11 Find the inverse of $f(x) = 2x + 3$

The graph of f^{-1} is the **reflection of f in the line $y = x$** .

The domain of f is the range of f^{-1} , and the range of f is the domain of f^{-1} .

An inverse function exists only if f is **one-to-one**. $f(x) = x^2$ has no inverse on $\mathbb{R}$.

Restrict the domain to $x \geq 0$ and the inverse is $\sqrt{x}$.

Extended: composite functions

A **composite function** applies one function after another. $(f \circ g)(x) = f(g(x))$.

Do **g first**, then f .

EXAMPLE $f(x) = 2x + 1$, $g(x) = x^2$

EXAMPLE $(f \circ g)(x) = f(x^2) = 2x^2 + 1$

The order matters.

EXAMPLE $(g \circ f)(x) = g(2x + 1) = (2x + 1)^2 = 4x^2 + 4x + 1$

EXAMPLE This is not the same as $f(g(x))$

CHECK 12 With $f(x) = x + 5$ and $g(x) = 3x$, find $(f \circ g)(x)$

To evaluate at a number, work from the inside out.

EXAMPLE $(f \circ g)(3) = f(g(3)) = f(9) = 19$

To solve a composite equation, simplify first.

EXAMPLE $f(g(x)) = 19$

EXAMPLE $2x^2 + 1 = 19$

EXAMPLE $x^2 = 9$

EXAMPLE $x = 3$ or $x = -3$

CHECK 13 Find $(f \circ f)(x)$ when $f(x) = 2x + 1$

A function composed with its inverse gives x .

EXAMPLE $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Using functions to model

A taxi charges CHF 5 plus CHF 2 per kilometre.

EXAMPLE $C(d) = 5 + 2d$

EXAMPLE Domain $d \geq 0$

EXAMPLE Range $C \geq 5$

Interpret the answer in context.

EXAMPLE $C(d) = 21$ gives $5 + 2d = 21$

EXAMPLE $d = 8$ so the journey is 8 km

CHECK 14 What does $C(10)$ mean?

The domain in a real problem may be limited.

A length cannot be negative and a number of people must be a whole number.

Common mistakes

×

“ $f(x) = 2x + 1$ means $f \times x$.”

✓

$f(x)$ is notation, read “ f of x ”. It is the output for input x : $f(3) = 7$.

×

“ $f^{-1}(x) =$

$$\frac{1}{f(x)}$$

.”

✓

The inverse undoes the function. The reciprocal is

$$\frac{1}{f(x)}$$

. They are different.

×

“For $f(x) = x^2 + 1$, $f(-3) = -9 + 1 = -8$.”

✓

Square the bracket: $(-3)^2 = 9$, so $f(-3) = 10$.

×

“ $(f \circ g)(x)$ means do f first.”

✓

Do g first: $(f \circ g)(x) = f(g(x))$.

×

“Domain is the y values and range is the x values.”

✓

Domain is the inputs (x). Range is the outputs (y).

×

“ $-2x = 6$, so $x = 3$.”

✓

Divide both sides by -2 : $x = -3$.

×

“The range of $f(x) = x^2$ is all real numbers.”

✓

Squares are never negative: the range is $\{y \in \mathbb{R} : y \geq 0\}$.

×

“ $\{x \in \mathbb{R} : x > 2\}$ includes 2.”

✓

A strict inequality leaves the end value out. Use $\geq$ to include 2.

Answers to the checks

Cover this page until you have tried the check yourself.

Check	Answer
Check 1	Solve $4x - 7 = 21$ Answer: $4x = 28$, so $x = 7$.
Check 2	Solve $3(x + 2) = 21$ Answer: $x + 2 = 7$, so $x = 5$.

Check	Answer
Check 3	Solve $7x + 1 = 3x + 13$ Answer: $4x = 12$, so $x = 3$.
Check 4	Solve $\frac{x}{3} - 2 = 4$ Answer: $\frac{x}{3} = 6$, so $x = 18$.
Check 5	Is 4 in $\{x \in \mathbb{R} : x \geq 4\}$? Answer: Yes, because $\geq$ includes 4.
Check 6	Is $\{(2, 3), (4, 3), (5, 1)\}$ a function? Answer: Yes. Each input has one output.
Check 7	Find the range of $f(x) = x + 4$ for $x \in \{1, 2, 3\}$ Answer: $\{5, 6, 7\}$
Check 8	State the domain of $f(x) = \frac{1}{x - 3}$ Answer: $\{x \in \mathbb{R} : x \neq 3\}$
Check 9	If $f(x) = 5 - 2x$, find $f(3)$ Answer: $5 - 6 = -1$
Check 10	If $f(x) = 2x + 3$, solve $f(x) = 11$ Answer: $2x + 3 = 11$, so $x = 4$.
Check 11	Find the inverse of $f(x) = 2x + 3$ Answer: $x = 2y + 3$, so $f^{-1}(x) = \frac{x - 3}{2}$
Check 12	With $f(x) = x + 5$ and $g(x) = 3x$, find $(f \circ g)(x)$ Answer: $f(3x) = 3x + 5$
Check 13	Find $(f \circ f)(x)$ when $f(x) = 2x + 1$ Answer: $2(2x + 1) + 1 = 4x + 3$
Check 14	What does $C(10)$ mean? Answer: The cost of a 10 km journey: $5 + 20 = \text{CHF } 25$.